Exact Treatment of Pure Coulomb Distortions in (X,n) Stripping Reactions*

FERNANDO B. MORINIGO
California Institute of Technology, Pasadena, California

(Received 13 December 1963; revised manuscript received 24 February 1964)

We obtain simple analytic formulas to express the amplitudes for stripping a particle into an orbit of arbitrary L value, using a Coulomb wave to describe the relative motion of the incident particles and a plane wave to describe the relative motion of the products. This approximation may be appropriate for the description of (X,n) reactions, where X is any charged projectile, and n is a neutron, particularly when the incident energy is well below the Coulomb barrier. Initially, the neutron is assumed to be bound to a particle which is later captured by the target; the wave function of this initial bound state is taken to be asymptotic and of zero orbital angular momentum, of the form $e^{-\alpha r}/r$. The resulting nucleus is described as a bound state of two particles moving with arbitrary relative orbital angular momentum L; the radial wave function of this bound state may be taken to be of the form $r^{L-1}(e^{-\beta r}-e^{-\beta' r})$. The cross sections predicted by these amplitudes are compared to the cross sections predicted by the analogous plane-wave Born approximation, and graphs are shown for a representative case. The qualitative appearance of the angular distribution is found to be much the same in both cases; however, the Coulomb-wave calculation predicts cross sections of smaller magnitude with previously assigned values of the reduced widths.

I. INTRODUCTION

In a previous paper[1] we have obtained convenient analytic expressions to represent the amplitudes of stripping reactions which occur by capture of a particle into an orbit of angular momentum $L=0$, using a Coulomb wave for the description of the relative motion of the incident particles, and a plane wave for the relative motion of the products. In this paper, we present an analogous treatment appropriate when the particle is captured into an orbit of arbitrary angular momentum L, and we include specifically the possibility that the particles involved have spins. The calculation of analytic expressions including the Coulomb distortions in both entrance and exit channels may be carried out with analogous methods[2] but the mathematical handling is considerably more involved, and the amplitudes cannot be expressed in terms of elementary functions; such calculations are the subject of a paper now in preparation.

The aim has been to develop expressions for stripping amplitudes which should be nearly as easy to understand and use as plane-wave expressions, yet have the enormous advantage of not ignoring the Coulomb distortions. Heretofore it has not been possible to take into account Coulomb distortions without going into a full optical-model treatment.[3] Our theory includes that distortion which undoubtedly dominates the behavior of cross sections at bombarding energies below the Coulomb barrier not near a resonance, and introduces no free parameters. It is a first-order Born approximation; it does not differ from the simplest plane-wave treatments except in the use of a Coulomb wave instead of a plane wave. Because the analytic expressions obtained involve only elementary functions and have a manageable simplicity, the comparison between the theoretical predictions and experimental results can be made very straightforwardly.

Our analytic expressions for the amplitudes result from making the following approximations:

(1) Initially the neutron to be emitted is in a bound state of $L=0$, whose radial wave function is described by an asymptotic form $e^{-\alpha r}/r$, where α is the wave number related to the separation energy of the neutron.

(2) The interaction of the emitted neutron with the target is neglected.

(3) In the initial state, we neglect all reaction waves; we describe it as a pure Coulomb scattering of target and projectile.

(4) The resultant nuclear state is described by a two-body wave function having a unique value of the orbital angular momentum, that is, it has an angular wave function $Y_L^M(\Omega)$; its radial dependence is taken to be $r^{L-1}e^{-\beta r}$.

These approximations lead to our particular form for the transition amplitude. Its main features are conveniently summarized in a diagram such as that of Fig. 1, where particle A represents the neutron. The description of the process is made in terms of three particles, A, B, and C. The reaction proceeds through the exchange of the particle B. The interactions of A with B, and of B with C, are treated exactly in terms of the postulated normalized wave functions and of reduced widths G_{AB}^0 and G_{BC}^f which appear as coefficients in the amplitude. In our treatment, we assume that the relative motion in the incident channel is a pure Coulomb wave, which is something like including all possible exchanges of photons between the incident particles.

* Supported by the U. S. Office of Naval Research.

[1] F. B. Morinigo, Phys. Rev. **133**, B65 (1964).

[2] The special case of $L=0$ has been treated in different ways by K. A. Ter-Martirosian, Zh. Eksperim. i Teor. Fiz. **29**, 713 (1955) [English transl.: Soviet Phys.—JETP **2**, 620 (1956)] and F. B. Morinigo, Nucl. Phys. **50**, 136 (1963).

[3] A good account of the status of direct interactions in 1961 and 1962 is available in the papers presented at the Manchester Conference, *Proceedings of the Rutherford Jubilee International Conference*, edited by J. B. Birks (Heywood & Co. Ltd., London, 1961) and at the Padua Conference, E. Clementel and C. Villi, *Direct Reactions and Nuclear Reaction Mechanisms* (Gordon and Breach, Inc., New York, 1963).

It has been possible to arrive at analytic expressions of manageable complexity because of two reasons. Firstly, in the computations of the amplitudes we extend the radial integrations all the way to the origin instead of having a cutoff at the nuclear surface. This makes it feasible to use the parabolic type expressions for the Coulomb wave; we eventually arrive at expressions which contain all the partial waves together, and thus we eliminate the summations over an infinite number of partial waves which would be required if we used spherical Coulomb waves. Secondly, we have chosen the particular form $r^{L-1}e^{-\beta r}$ to represent the radial wave function of the final bound state. This permits a key intermediate simplification in the course of the computation. This form is in itself suitable for representing bound states among the light nuclei. In any case, since the answers obtained are analytic, the amplitudes corresponding to wave function forms containing the higher powers of r may be generated by taking derivatives of our answer with respect to the parameter β.

II. DESCRIPTION OF THE FINAL BOUND STATE

Within the framework of a nonrelativistic treatment, the final bound state should presumably be described by a function which is the solution of a Schrödinger equation. Our form $Y_L{}^M(\Omega)r^{L-1}e^{-\beta r}$ corresponds to a potential which bears no particular resemblance to those used in nuclear theory. Thus, it might perhaps be best to look upon β as a variational parameter which is to be adjusted so as to make $Y_L{}^M(\Omega)r^{L-1}e^{-\beta r}$ correspond as closely as possible to the wave function itself. Since $1/\beta$ is the characteristic dimension of the nucleus represented, it is very unlikely that the best value of β will be greatly different from the bound-state wave number, that is, $\hbar^2\beta^2\cong 2mE$, where m is the reduced mass and E is the separation energy.

As an estimate of how well the trial functions can represent other wave functions, we may present the result corresponding to assumed "true" wave functions of the same form as the hydrogen wave functions of lowest radial quantum number. These functions are the following:

$$\chi(L,M,r,\theta,\varphi)=Y_L{}^M(\theta,\varphi)r^Le^{-\beta_0 r}\times[(2\beta_0)^{2L+3}/(2L+2)!]^{1/2}. \quad (2.1)$$

For applications among light nuclei, states having radial nodes would ordinarily be so high in excitation that stripping theory need not consider them. The normalized trial functions corresponding to a parameter β are

$$\chi_t(L,M,r,\theta,\varphi,\beta)^*=Y_L{}^M(\theta,\varphi)^*r^{L-1}e^{-\beta r}\times[(2\beta)^{2L+1}/(2L)!]^{1/2}. \quad (2.2)$$

The overlap of this trial function with the hydrogen-type wave function is easily computed. If we choose to express its value in terms of the ratio $x=\beta/\beta_0$, where $\hbar^2\beta_0{}^2=2mE$ exactly, we find that the maximum overlap occurs when

$$x_{\max}=(2L+1)/(2L+3), \quad (2.3a)$$

$$\langle\chi_t|\chi\rangle_{\max}=(2L+1)^{L+1}(2L+3)^{L+3/2}/(2L+2)^{2L+5/2}. \quad (2.3b)$$

The maximum overlap is worst for $L=0$; the numerical values for $L=0$, 1, 2 are 0.92, 0.98, 0.99. Analogous estimates for the overlap of our trial functions with wave functions corresponding to potentials of the square well or harmonic oscillator or Hulthén shapes yield maximum values of the overlap all in the same range. Thus, we may conclude that the single-parameter description of the final bound state may be very suitable, especially for the values of L greater than zero.

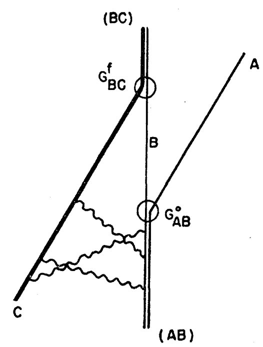

FIG. 1. Diagram illustrating the interactions which are included in our calculation. The relative motion in the incident channel is described by a Coulomb wave, which is symbolized by exchanges of photons. The small circles represent interactions which are treated exactly, in terms of a normalized wave function and a reduced width.

We may note that a radial dependence such as $r^{L-1}(e^{-\beta r}-e^{-\beta' r})$ may be considered by simply taking differences of amplitudes corresponding to two decay parameters, β and β', and making a suitable change in the normalization constant. It is clear that by using these two variational parameters β and β', any reasonably smooth radial function having no nodes might be represented with excellent accuracy. For applications among heavier nuclei, wave functions with radial nodes may be constructed by taking suitable linear combinations of terms of the same form.

III. EVALUATION OF THE TRANSITION AMPLITUDES FOR SPINLESS PARTICLES

If the incident and final relative motions asymptotically tend to plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}$ and $e^{i\mathbf{k}_f\cdot\mathbf{r}}$, the cross section for a reaction is given by

$$(d\sigma/d\Omega)=(m_0m_f/4\pi^2\hbar^4)(k_f/k)|T_{f0}|^2, \quad (3.1)$$

where m_0,m_f are the reduced masses of the relative motions, $\mathbf{k}$ and $\mathbf{k}_f$ are the initial and final wave vectors, and T_{f0} is the transition amplitude. In our treatment, the following formal expression is the result of the assumptions 1, 2, and 3 of Sec. I:

$$T_{f0}=(-2\pi\hbar^2N_0/m_{AB})\times\int e^{-i\mathbf{K}'\cdot\mathbf{r}}X_f(-\mathbf{r})^*\varphi_{\mathbf{k}}^+(\mathbf{r})r^2drd\Omega\,, \quad (3.2)$$

which is deduced as Eq. (2.8) of Ref. 1; N_0 is the constant which normalizes the initial bound state, m_{AB} is the reduced mass of the initial bound system (AB), $X_f(\mathbf{r})$ is the wave function of the final bound state, $\mathbf{K}'=\mathbf{k}_fm_C/(m_C+m_B)$, and $\varphi_{\mathbf{k}}^+(\mathbf{r})$ is the initial scattering state. We shall use for $\varphi_{\mathbf{k}}^+$ a Coulomb wave which asymptotically tends to the incident plane wave, plus outgoing scattered waves. We first do the calculation as though all particles were spinless, and later quote results in the general case of arbitrary spins. To represent the initial and final bound states, we use the following:

$$X_f(\mathbf{r})=N_fr^{L-1}e^{-\beta r}Y_L{}^M(\Omega)\,,\qquad X_0(\mathbf{r})=N_0e^{-\alpha r}/r\,, \quad (3.3a)$$

$$N_f{}^2=(G_{BC}{}^f)^2(2\beta)^{2L+1}/(2L)!\,,\qquad N_0{}^2=(G_{AB}{}^0)^2(2\alpha)/4\pi\,. \quad (3.3b)$$

The parameter α is given by $\alpha^2=2m_{AB}E_{AB}/\hbar^2$, where E_{AB} is the separation energy, the energy required to separate the initial state into particles A and B. We note that $X_f(-\mathbf{r})=(-)^LX_f(\mathbf{r})$. The amplitude for a transition to a particular magnetic substate $\langle L,M|$ consists of an energy-dependent factor D, which contains kinematic dependences and the Coulomb parameter n, and a space integral which contains all the angular dependences

$$T_{f0}(L,M,\beta)=D\times I(\beta,k,K',Q,n,L,M)\,, \quad (3.4a)$$

$$D=(-)^{L+1}(2\pi N_0N_f\hbar^2/m_{AB})\times\Gamma(1+in)e^{-n\pi/2}\,, \quad (3.4b)$$

$$n=Z_CZ_{AB}e^2m_0/\hbar^2k\,, \quad (3.4c)$$

$$I(\beta,k,K',\mathbf{Q},n,L,M)=\int\exp(i\mathbf{Q}\cdot\mathbf{r})\,Y_L{}^M(\mathbf{r})^*r^{L+1}e^{-\beta r}\times F(-in,1,ikr-i\mathbf{k}\cdot\mathbf{r})drd\Omega\,, \quad (3.4d)$$

$$\mathbf{Q}=-\mathbf{K}'+\mathbf{k}\,, \quad (3.4e)$$

where eZ_{AB}, eZ_C are the charges of target and projectile, and $\mathbf{Q}$ is the vector representing the change in linear momentum of the particle C; it plays the role of the "momentum transfer" in the plane-wave theory. We may carry out the space integrations indicated in Eq. (3.4d) after introducing an integral representation of the confluent hypergeometric function[1,4]

$$F(b,c,z)=[\Gamma(c)/\Gamma(b)\Gamma(c-b)]\times\int_0^1 e^{zt}t^{b-1}(1-t)^{c-b-1}dt\,, \quad (3.5)$$

with $c=1$, $b=-in$, $z=ikr-i\mathbf{k}\cdot\mathbf{r}$. The space integrations are easily carried out after expanding in spherical harmonics and spherical Bessel functions the factor $e^{i\mathbf{q}\cdot\mathbf{r}}$, where $\mathbf{q}=\mathbf{Q}-\mathbf{k}t$. To do the angular integration we use the orthogonality of the spherical harmonics; the remaining radial integral is expressible[4] in terms of a hypergeometric function times other factors. Our choice of the form for the bound-state function makes one index of the hypergeometric function identically zero, hence the result is particularly simple. The expression which remains to be integrated over the parameter t is

$$I(\beta,k,K',Q,n,L,M)=D_0[\Gamma(c)/\Gamma(b)\Gamma(c-b)]\times\int_0^1 q^L[q^2+(\beta-ikt)^2]^{-L-1}\times Y_L{}^M(\mathbf{q})t^{b-1}(1-t)^{c-b-1}dt\,, \quad (3.6a)$$

where

$$D_0=(4\pi)i^L\Gamma(\tfrac{1}{2})\Gamma(2L+2)/[2^{L+1}\Gamma(L+\tfrac{1}{2})]\,. \quad (3.6b)$$

The argument of the spherical harmonic appearing in Eq. (3.6a) is understood to represent the direction of the vector $\mathbf{q}$ relative to an arbitrary reference axis, whose orientation we are still free to choose. The most explicit comparison with the plane-wave theory can be made if we choose our axis of reference to lie along the direction of the incident wave vector $\mathbf{k}$. The spherical harmonics referred to this axis [which we denote by $Y_L{}^M(\mathbf{q};\mathbf{k})$] may be expressed in terms of an azimuthal angle and a homogeneous polynomial of order L in products of $\sin\gamma$ and $\cos\gamma$, where γ is the angle between the vectors; the sine and cosine may be calculated from the definitions of the vectors as follows:

$$\cos\gamma=(\mathbf{k}\cdot\mathbf{q}/kq)=(Q/q)\cos\theta[1-(k^2/\mathbf{k}\cdot\mathbf{Q})t]\,, \quad (3.7a)$$

$$\sin\gamma=[1-\cos^2\gamma]^{1/2}=(Q/q)\sin\theta\,, \quad (3.7b)$$

where θ is the angle between the vectors $\mathbf{Q}$ and $\mathbf{k}$. When we insert the expressions (3.7a,b) into formulas for the spherical harmonics appearing in the integral (3.6a), the factor q^L is canceled out, and effectively replaced by Q^L, which is independent of the parameter t. At the same time, $\sin\gamma$ is replaced by $\sin\theta$, which is also independent of t. Since it is possible to express $Y_L{}^M(\mathbf{q};\mathbf{k})$ in such a way that at most $L-|M|$ powers

[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 605.

of $\cos\gamma$ are required,[5] the dependence of the integrand of Eq. (3.6a) on the parameter t may be simplified considerably. An expression valid for $L=0$, 1, 2 is

$$I(\beta,k,K',Q,n,L,M)=I_0(L)Y_L{}^M(\mathbf{Q};\mathbf{k})$$
$$\times[\Gamma(c)/\Gamma(b)\Gamma(c-b)]\int_0^1 t^{b-1}(1-t)^{c-b-1}$$
$$\times(1-G_0t)^{-L-1}\{1-\lambda t+\mu t^2\}dt\,, \quad (3.8a)$$

where

$$I_0(L)=D_0Q^L/(Q^2+\beta^2)^{L+1}\,, \quad (3.8b)$$
$$G_0=(2i\beta k+2\mathbf{k}\cdot\mathbf{Q})/(Q^2+\beta^2)\,, \quad (3.8c)$$
$$\mu=2\epsilon^2\cos^2\theta\delta(|M|,2)/(3\cos^2\theta-1)\,,$$
$$\epsilon=k^2/\mathbf{k}\cdot\mathbf{Q}\,, \quad (3.8d)$$
$$\lambda=\epsilon\delta(|M|,1)+2\mu/\epsilon.$$

The remaining integrals over the parameter t may now be recognized as being representations of the hypergeometric function.[4] The final result is

$$\begin{aligned}I(\beta,k,K',Q,n,L,M)&\\ =I_0(L)Y_L{}^M(\mathbf{Q};\mathbf{k})\{&F(L+1,\,-in,\,1,\,G_0)\\ &-(-in)\lambda F(L+1,\,-in+1,\,2,\,G_0)\\ &+\tfrac{1}{2}(-in)(-in+1)\mu F(L+1,\,-in+2,\,3,\,G_0)\}.\end{aligned} \quad (3.9)$$

For the cases of interest, L is a positive integer or zero. The factor involving the hypergeometric functions can in these cases be expressed entirely in terms of elementary functions, and it may be seen that it leads to a rather weak angular dependence. The quantity in braces in Eq. (3.9) may be rewritten as

$$(1-G_0)^{in}\cdot R(L,M)\,,$$

where

$$\begin{aligned}R(L,M)=[&F(-L,\,-in,\,1,\,H_0)\\ &+(in)(1-G_0)^{-1}\lambda F(-L+1,\,-in+1,\,2,\,H_0)\\ &+\tfrac{1}{2}(-in)(-in+1)(1-G_0)^{-2}\\ &\qquad\times\mu F(-L+2,\,-in+2,\,3,\,H_0)]\end{aligned}$$

and

$$H_0=G_0/(G_0-1)\,. \quad (3.10)$$

Since the first index of the hypergeometric functions is a negative integer, the expansion[4] in powers of H_0 contains a finite number of terms, $(L+1)$ from the first, or (L) from the second. Putting together Eqs. (3.4a,b), (3.9), (3.10), (3.8c), and (3.6b), we have our explicit expression for the transition amplitude T_{f0}. The plane-wave theory is the special case obtained in the limit that $n\rightarrow 0$; the factors $R(L,M)$, $\Gamma(1+in)$ and $e^{-n\pi/2}$ are replaced by unity in this limit.

[5] This is obvious by inspection of a list of the spherical harmonics, as given by A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), or, for $M>0$ by successive application of the lowering operator $L_-=e^{-i\varphi}[\partial/\partial\theta+i\cot\theta\partial/\partial\varphi]$ on the function $Y_L{}^L(\theta,\varphi)$, which is well known to be proportional to $(\sin\theta)^Le^{iL\varphi}$. For $M<0$, we may use the symmetry rule $Y_L{}^{-M}=(-)^MY_L{}^{M*}$.

IV. DIFFERENTIAL CROSS SECTIONS IN COULOMB-WAVE AND PLANE-WAVE THEORIES

The usual experiment does not measure the spin polarization of the nuclei; in the plane-wave theory (PWBA) with spinless particles, a summation of the transition rates to all the final magnetic substates $\langle L,M|$ leads to the following expression for the cross section:

$$(d\sigma/d\Omega)_{\text{PWBA}}=W[Q^L/(Q^2+\beta^2)^{L+1}]^2(2L+1)/4\pi\,, \quad (4.1a)$$
$$\begin{aligned}W=&(m_0m_f/m_{AB}{}^2)(k_f/k)\\ &\times(G_{BC}{}^fG_{AB}{}^0)^2(2\beta)^{2L+1}(2\alpha)(4\pi)\\ &\times[\Gamma(\tfrac{1}{2})\Gamma(2L+2)/2^{L+1}\Gamma(L+\tfrac{1}{2})]^2.\end{aligned} \quad (4.1b)$$

In the Coulomb-wave theory (CWBA), the summation over the final M values cannot be carried out by manipulations; we must have a specific expression or an evaluation of the transition rates to each magnetic substate. The structure of the answer is the same as that of Eq. (4.1); the only change is that the Coulomb penetration factors appear, and the sum over M replaces the factor $(2L+1)/4\pi$. The cross section is

$$\begin{aligned}(d\sigma/d\Omega)_{\text{CWBA}}=&(d\sigma/d\Omega)_{\text{PWBA}}\\ &\times[2\pi n/(e^{2\pi n}-1)][4\pi/(2L+1)]\\ &\times\exp\{2n\arctan[2\beta k/(K'-k^2+\beta^2)]\}R_0\,,\end{aligned} \quad (4.2a)$$

where

$$R_0=\textstyle\sum_M|R(L,M)|^2. \quad (4.2b)$$

The arctangent is to be chosen so as to lie between 0 and π. The only new angular dependence comes from the factor R_0, which is the sum of absolute squares of polynomials having at most L powers of H_0. In terms of the wave numbers and the angle θ, the quantity H_0 is

$$H_0=2(k^2-kK'\cos\theta+i\beta k)/(\beta^2+K'-k^2-2i\beta k)\,. \quad (4.3)$$

The dependence of this quantity on the angle θ is not particularly strong; hence, we may conclude that it is unlikely that the Coulomb distortion will alter the appearance of the angular distributions to something radically different from the predictions of the plane-wave theory.

Now, since the only neutral particle in nuclear physics is the neutron, which has a spin, we must carry out a calculation for particles with spin before having a theory applicable to cases of interest. The particles A, B, and C are assumed to be endowed with spins J_A, J_B, and J_C. The spins J_A and J_B are coupled (with $L=0$) to form the initial state, of spin J_0. We let the spin of the final state be J_f; we assume it is formed from J_B, J_C, and L by first coupling J_B and L to form an intermediate j, which then couples to J_C to give J_f. The only change when we sum over final projections and average over initial projection occurs in Eq. (4.1), which

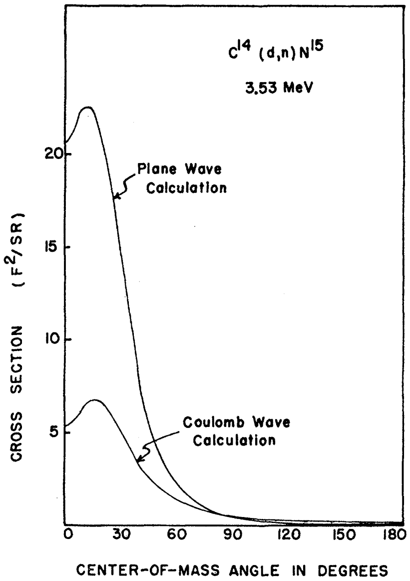

FIG. 2. The differential cross sections predicted by the plane-wave theory and the Coulomb-wave theory for the case of the reaction $C^{14}(d,n)N^{15}$ at 3.53-MeV bombarding energy, assuming $L=1$, for the case that the reduced widths take on their maximum possible value of unity, as explained in Sec. IV.

must be modified by multiplying it by a statistical factor

$$[(2J_0+1)(2J_f+1)/(2J_B+1)(2L+1)] \times[1/(2J_C+1)(2J_0+1)]. \quad (4.4)$$

It is possible to use the same mathematics to generate analytic expressions in the approximation that the final state is described by a two-parameter wave function, of radial dependence $r^{L-1}(e^{-\beta r}-e^{-\beta' r})$; all that we need to do is to use a difference of terms of the form of Eq. (3.9) in the new formula for $R(L,M)$. This procedure will in most cases have a negligible effect on the predicted angular distributions, because the new term modifies only the contribution of the integral (3.4d) in a region near the origin, which is suppressed both by the weighing factor r^{L+1}, and by the Coulomb repulsion. The only significant change occurs in the normalization factor, so that a somewhat larger value of the reduced width corresponds to a cross section of the same magnitude.

We have carried out a numerical evaluation of the cross sections predicted by Eqs. (4.1a) and (4.2a) for the case of the reaction $C^{14}(d,n)N^{15}$ (ground state) at 3.53-MeV bombarding energy, which is exothermic with an energy release of 7.987 MeV and appears to proceed by an $L=1$ capture.[6] In Fig. 2 we show the theoretical curves predicted by the expressions (4.1a) and (4.2a) for the case that the reduced widths take their maximum value of unity, and with a decay parameter $\beta=\frac{1}{2}\beta_0$.

V. CONCLUSIONS

The preceding discussion and example indicate that the Coulomb-wave expressions obtained in this paper predict the qualitative features of the angular distributions to be little different from those of the plane-wave theory; it may thus be expected that our formulas will prove satisfactory for the fitting of many angular distributions. Thus, it is shown that the recurrent puzzle of having obtained good fits with plane-wave expressions is not a mystery, but to be expected to the extent that it is the Coulomb repulsion which dominates the distortions. This is the same conclusion obtained in our previous paper[1] dealing with the $L=0$ case only. The chief difference between plane-wave and Coulomb-wave theory lies in the magnitude predicted, if we assume a given value of the reduced widths. The Coulomb-wave theory predicts cross sections of smaller magnitude; conversely, if we obtain reduced widths by fitting the data, the Coulomb-wave theory will yield larger values for the reduced widths. This is in the direction of better agreement with shell-model theory and with other experimental evidence.[6] It will undoubtedly be of great interest to carry out a fit with these expressions to a large variety of experimental data, in order to test the general usefulness of the method. A future comparison with analogous DWBA calculations should also be of interest.

ACKNOWLEDGMENT

The author is grateful to Dr. J. Eichler for having guided him to some of the earlier work on the subject, and for his many valuable comments.

[6] M. H. Macfarlane and J. B. French, Rev. Mod. Phys. **32**, 657 (1960).